01.00.00 Physico-mathematical sciences

01.00.00 Физико-математические науки

UDC 530

The Development of the Proving Process Within a Dynamic Geometry Environment

Danh Nam Nguyen

Thai Nguyen University of Education, Vietnam
E-mail: danhnam.nguyen@dhsptn.edu.vn

Abstract. In this paper we classify student's proving level and design an interactive help system (IHS) corresponding with these levels in order to investigate the development of the proving process within a dynamic geometry environment. This help system was also used to provide tertiary students with a strategy for proving and to improve their proving levels. The open-ended questions and explorative tasks in the IHS make a contribution to support students' learning of proving, especially during the processes of realizing invariants, formulating conjectures, producing arguments, and writing proofs. This research wants to react on the well-known students' difficulties in writing a formal proof. The hypothesis of this work is that these difficulties are based on the lack of students' understanding the relationship between argumentation and proof. Therefore, we used Toulmin model to analyze student's argumentation structure and examine the role of abduction in writing a deductive proof. Furthermore, this paper also provides mathematics teachers with three basic conditions for understanding the development of the proving process and teaching strategies for assisting their students in constructing formal proofs.

Keywords: Proof; proving process; proving level; interactive help system; Toulmin model; argumentation; abduction; visual thinking; geometric invariant.

1. Introduction

Proving is a crucial activity within mathematical classrooms at the different educational levels. It provides a way of thinking that deepens mathematical understanding, broadens the nature of human reasoning and fosters students' creativity. Polya (1954) has claimed that understanding is a necessary condition for proving because when students have reassured themselves that a theorem is true, they will start proving it. The NCTM standards (2000) also emphasized in a particular section the importance of developing students' reasoning and proving abilities, formulating conjectures, producing arguments, and using various methods of approaching proofs. However, Mariotti (2007) has argued that teaching students the key ideas of proofs is not an easy task. Therefore, mathematics teachers are usually faced with the difficult task of teaching students how to understand the proving process in mathematics classroom. Therefore, understanding the development of the proving process may contribute in gaining insight into the understanding of the invention of mathematical ideas and the nature of proofs. That is the reason why tertiary students should learn how to write, read, understand, and construct proofs, even though the functions of proofs are not fulfilled in the teaching of proofs in schools and remain hidden in some mathematics textbooks (see e.g. Hanna, 2000; de Villiers, 2003). In order to offer a situation for the construction of a formal proof, Edwards (1997) proposed the term "conceptual territory before proof". It was defined by demonstrating that conjecturing, reasoning, exploration, explanation, and validation constitute the essential elements before a proof. Thus, in this paper, we will propose a methodological model concerning this area in order to support students in constructing each element before writing a formal proof. As a result, we designed an IHS with the purpose of supporting students in constructing figures or diagrams, realizing geometric invariants, formulating conjectures, producing arguments, validating conjectures, and writing a formal proof.

Working within a dynamic geometry environment, such as GeoGebra, students would gain their understanding through verifying conjectures and transform understanding into an explanation as to "why" the statement is valid. Therefore, the IHS was designed to provide students with *open-ended questions* and *explorative tasks* so that they can construct proofs on their own.

Simultaneously, this help system could also develop a sense of proof and improve their geometric intuition during the proving process. In the mathematics teacher training universities, it is important to improve students' proving skills within a dynamic geometry environment. These prospective teachers also need to understand the development of the proving process in order to provide their students at a secondary school with a suitable strategy in approaching proofs. This means that the teachers should learn how to design instructional interventions to support their students in overcoming cognitive difficulties, enhancing proof techniques and strategies, and properly understanding mathematical proofs. For that reason, the IHS should also provide tertiary students with some strategies to bridge the cognitive gaps between the different phases of the proving process. Specifically, some task-based activities were designed to encourage students to produce arguments and write formal proofs. We also provided students with opportunities to think visually and dynamically, to look for geometric invariants, to formulate conjectures, to produce arguments, and to pose and answer the explorative questions on their own. A dynamic geometry environment, such as GeoGebra (e.g. Hohenwarter & Jones, 2007), can serve as a context for realizing geometric invariants, formulating conjectures about geometric objects and consequently lead to proof-generating situations. In particular, the dragging mode can play the role of a mediator in the transition from argumentation to proof. We also utilized the concept of cognitive unity* to reveal students' difficulties in bridging the gap between conjecture and argumentation, also the gap between argumentation and proof. At the end of the experimental teaching, we have also evaluated the effects of the IHS on the improvement of proving levels and the development of geometric thinking. In particular, we also studied the influence of dynamic visual thinking on enhancing students' geometrical intuition and revealing geometric invariants. Furthermore, we investigated the discussion among students while they used the IHS to support proof-related problems. Through group discussion, we analyzed the students' structure of argumentation and examined the role of abduction during the process of realizing geometric invariants and writing a formal proof.

2. Designing an interactive help system

Balacheff (1998) classified four levels of a proof to understand students' cognitive development in writing proofs such as naive empiricism, crucial experiment, generic example, and thought experiment. Students' approaches and methods in writing proofs could be analyzed and compared with the proof levels they have attained. However, these levels are not enough to determine how students understand the development of the proving process. For that reason, in this research, we classified seven proving levels that represent the developmental phases in the proving process. These levels are proposed as follows: *information* (level 0), *construction* (level 1), *invariance* (level 2), *conjecture* (level 3), *argumentation* (level 4), *proof* (level 5), and *delving* (level 6). In the IHS, we designed opened-questions and explorative tasks corresponding with these proving levels. In addition, based on students' solution of tasks, we can estimate their knowledge understanding and proving abilities. In order to determine how effective a stimulating help is on the proving process, the following features of help were considered:

- *Should be suitable to student's understanding and proving level:* The question or task is not too easy, not too difficult, and stimulates the student's thinking.
- *Should be heuristic and motivated:* There should be a gap between the help and proof idea as well as the solution of the problem. If the help is understood, it gives the whole secret away, very little remains for the student to do. It should also motivate students to prove by using some stimulating questions and tasks.
- *Should be instructive and natural:* Student may perceive naturally the proof ideas through exploring the problem with help and get into the habit of using these methods. In other words, students may definitely profit (the solution of the problem and the problem-solving strategies as well) from this help system.

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* "During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organizing some of previously produced arguments according to a logical chain" (Boero et al., 1996, p.124). In other words, cognitive unity is a situation or phenomenon where some arguments, which are produced for the plausibility of the conjecture during the conjecture production phase, become ingredients for the construction of a proof.

During the process of designing the IHS, we always keep the well-known heuristic principles of Polya and van Hiele levels of geometric reasoning in the back of our mind. *An open-ended question* (to direct thought) is used to find geometric invariants and connect arguments forming a formal proof. *An explorative task* (to convey information) is used to help students explore the problem on their own. By answering open-ended questions or completing explorative tasks, the proof ideas may also emerge gradually. Each proof-related problem needs some auxiliary or supplementary elements such as auxiliary diagrams and lines, geometric invariants and transformations, student's prior knowledge, supporting theorems, rules of inference, etc. The specific goals of the IHS with respect to student's proving level are described as follows:

Level 0: Information

In order to prove a conjecture (or problem), students should grasp and understand all the information related to the problem. If they are lacking in understanding or in interest, they are not motivated and could not tackle successfully the problem. Therefore, the IHS should provide students with understandable information aimed to point out the principal parts of the problem, the unknown, the data, and the condition.

Level 1: Construction

Students should construct the figure on their own by using a dynamic geometry software such as GeoGebra. To attain this level, students need some basic construction skills with straightedge and compass. Thanks to construction functions of GeoGebra software, students can construct their drawings easily such as intersect of two objects, midpoint of a segment, a line through two points, parallel/perpendicular lines, angle bisector, perpendicular bisector, tangents, segment with given length, angle with given measure, polygon, circle, conic, etc. However, students often have some difficulties in constructing drawings because of lacking these basic construction skills.

Level 2: Invariance

At this level, students need apply invariance principle to search for geometric invariants supporting proving process. Some helpful questions can be used to support students throughout this phase: What property is preserved as dragging? Which figures do not change their shapes as moving? Which figures are congruent or similar as moving? When the students get stuck on a geometrical problem, they usually try to use familiar problem or to specialize it in some way and then look for something familiar. However, the IHS will help students to consider the problem from different aspects. It contains two stages of looking for geometric invariants. Firstly, the students guess the transformations appearing in the problem. These can be realized by some signs: there exist constant angles, constant distances, constant directions, equal distances, equal angles, equal figures, regular polygon, fixed lines, fixed points, parallel lines, and so on. Secondly, students need to find geometric invariants by using dragging modality. Invariants of geometric transformation may be: measurement of angle, length of segment, parallelism, concurrency, perpendicularity, betweenness, collinearity, ratio of two segments, shape of a figure, etc. In general, realizing geometric invariants by using dragging mode will provide students with more supportive data for the proving process.

Level 3: Conjecture

A conjecture is a statement strictly connected with an argumentation and a set of conceptions where the statement is potentially true because some conceptions allow the construction of an argumentation that justifies it. It is the postulation that something ought to be true or false. Conjectures often originate from experimentation, numerical investigations and measurements. During the process of formulating conjectures, students work with arguments to construct a proof. As mentioned above, after realizing geometric invariants, the IHS should provide students with an open-ended question so that they can make a conjecture with the support of some activities such as: measure the area of the figure; check the relation between two objects; calculate the angle measurement, the distance between two points; and check the locus of the moving objects, etc.

Level 4: Argumentation

At this level, the IHS supports students to organize produced plausible arguments by collecting and combining them in order to form proofs. The data which are not necessary for proof has been also reduced or re-organized. Argument is not really part of a proof, but is needed to produce it. For the teaching and learning purposes, argumentation is a fruitful means to control the validity of reasoning. There are two levels of argumentation: as part of the proving tasks, specially for producing and organizing arguments; and in discussing procedures, as a means to assimilate and master elements of proving process. Therefore, argumentation is one of the most important phases of the proving process and it provides valuable data for writing proofs at the next level.

Level 5: Proof

Based on produced arguments, at this level, the IHS guides students to write proofs. Students have to select some helpful arguments, connect them to form a chain of reasoning. The use of mathematical language and logical laws are essential for students in this phase of the proving process. Therefore, the IHS should give a rule of inference to connect arguments or suggest some open-ended questions aimed at producing deductive arguments. At this level, students will be guided to write their formal proof and the IHS take the responsibility for supporting students to overcome their difficulties in writing proofs.

Level 6: Delving

Delving into a problem by reconsidering, expanding the result, students could not only consolidate their knowledge but also develop their ability of solving problems. At this level, the IHS suggests students to use some mathematical thinking strategies in the process of delving such as: brainstorming, generalization, expansion, specialization, analogy, decomposing and recombining, etc. Delving into a problem also means that the students should try to make their proofs as simple as possible.

In general, each of abovementioned levels has its own role in the proving process. Some students can ignore one of these levels if the proof idea suddenly appears. Then students can jump and go straight to the solution of the problem. The development of the proving process is described as the following methodological model:

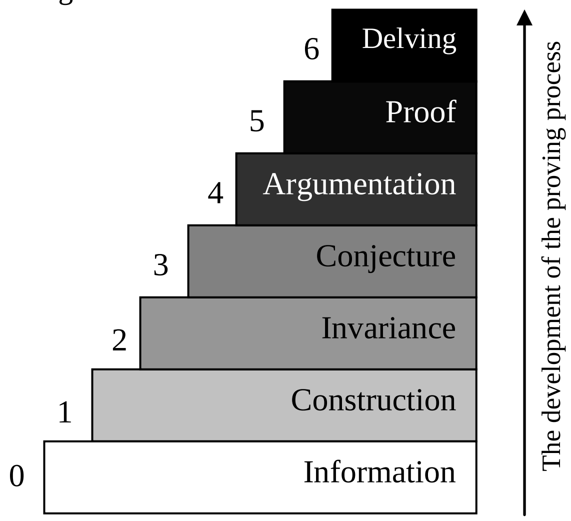


Fig. 1: A methodological model for understanding the proving process

Throughout the period of experimental teaching, we have realized that tertiary students spend almost all of their time in four phases in the development of the proving process such as *invariance*, *conjecture*, *argumentation*, and *proof*. In the invariance phase, students realize geometric invariants so as to generate the proof ideas. The ability of realizing these invariants depends upon student's proving level. In the transition from conjecture to argumentation phase, there is a cognitive gap between them (Pedemonste, 2001; 2007). Students must produce valuable arguments or construct a cognitive unity in the process of validating conjectures. Moreover, in the argumentation phase, students should use different kinds of inferences like induction, deduction, and abduction in which abduction plays a crucial role in explaining 'observed facts' and constructing proofs. It also makes a contribution to bridge the cognitive and structural gaps between argumentation and proof. Generally speaking, students have difficulties in writing a formal proof because of the existence of these gaps. Therefore, this paper also reveals the gaps and determines some fundamental aspects that influence on the proving process. These aspects are described in the following diagram:

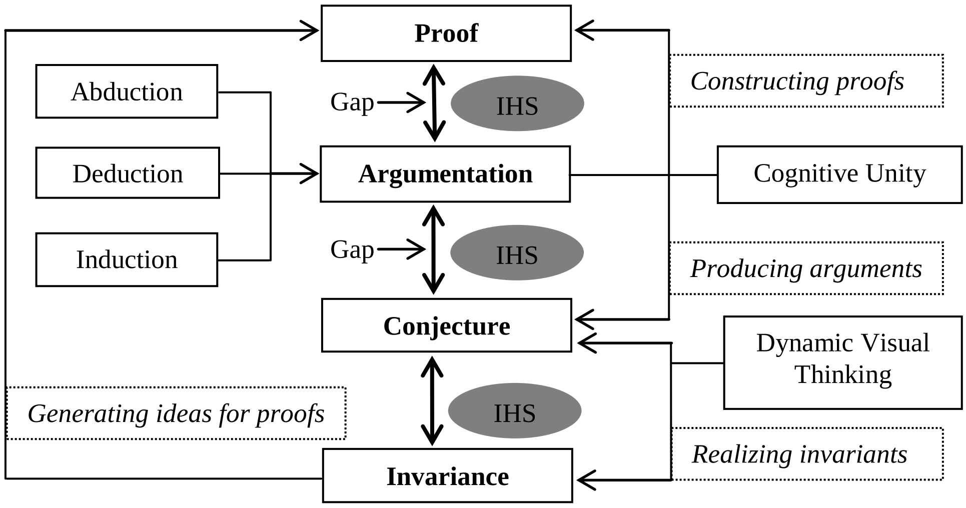


Fig. 2: Fundamental aspects that influence on the proving process

3. Research methodology

The data was collected during the summer semester 2010. The students were enrolled in a required elementary geometry classes for a teacher training course. The raw materials were firstly checked, coded, edited, entered into a computer, and subsequently analyzed. These materials consist of transcripts from the video and audio recording, students' worksheets, hypotheses, and teacher's field-notes. All of the students involved in our research were second-year students of Thai Nguyen University of Education in Vietnam. In a computer laboratory, students were divided into groups of three, who sat together at one computer. Each computer was installed with the GeoGebra software in order to create a dynamic environment for group-based activities. The reason for this division is the fact that working in groups positively affected the development of the proving process (e.g. Olivero, 2002). Especially, in the group-based activities, open-ended questions and explorative tasks in the IHS were sought out and were jointly considered. During the process of using the computer, groups of students drag the point, measure the length, check the relationship, and formulate conjectures. These activities could take a step toward addressing discussion and reasoning. Firstly, we observed students' behaviors associated with audio and snapshot video recordings. This method of observation is a powerful tool that offers us the chance to gather live data from the students' discussion, get inside situations and observe directly what is happening, thus collect more valid and authentic data. We also installed Wink® software on each computer in order to capture and audio-record of all the group discussions. Furthermore, we used teacher's field notes which recorded some remarkable activities and reasonable arguments that emerge from discussions in order to interpret the student's thinking and behavior during the process of constructing cognitive unity (Boero et al., 1996). Moreover, we formulated some hypotheses in order to answer the quantitative questions by testing hypotheses and analyzing correlation between variables. As a result, we investigated the cause-effect relationship between the use of the IHS during the proving process and students' test scores and proving abilities.

Toulmin (1958) argued that the abstract and formal criteria of mathematical logic have little applicability to the methods of assessing arguments. Therefore, he built a model to represent the structures of argumentation in different fields. Toulmin argued that in any argumentation the first step is expressed by a *claim* (an assertion, an opinion or a conjecture). The second step consists of the production of *data* supporting it. It is important to provide justification or warrant for using the concerned data as support for the data-claim relationships. The *warrant* can be expressed as a principle, a rule or a theorem. The warrant acts as a bridge which connects the data and the claim. Claim, data, and warrant form the basic structure of argumentation. This model was used to represent the structure of produced arguments during the proving process:

C (claim): the statement of the speaker

D (data): data justifying the claim **C**

W (warrant): the inference rule that allows data to be connected to the claim

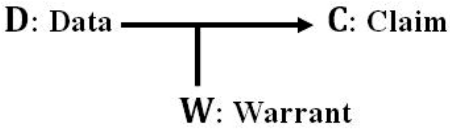


Fig. 3: Toulmin basic model of argumentation

The most important role of Toulmin model is to analyze the relationship between argumentation and proof, especially the gap between them. In Toulmin model, a step appears as a deductive structure because data and warrants lead to a claim. Therefore, it is useful to represent a chain of logical deduction. However, it is also a powerful tool to represent an abductive structure, which can be used to explicate the role of abduction in the proving process (Pedemonte & Reid, 2011). In particular, the students can reverse abductive structure in order to write a deductive proof and understand logical reasoning produced in the proving process. The term "abduction" was introduced by Peirce (1960) to differentiate this type of reasoning from deduction and induction. Abduction is an inference which allows the construction of a claim starting from an observed fact (see e.g. Peirce, 1960; Polya, 1962; Magnani, 2001). In other words, abduction plays the role of generating new ideas or hypotheses. At the tertiary level, students tend to use abduction in producing arguments and searching for proof ideas because the logic of abduction contributes to the conceptual understanding of a phenomenon. Pedemonte & Reid (2011) presented abduction when the arguer knows the rule of inference in Toulmin model as follows:

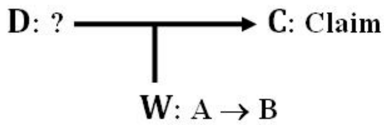


Fig. 4: Abduction in Toulmin model of argumentation

The term 'abductive argumentation' originated from abduction. It has been considered as a type of 'backwards' reasoning and as an 'inference to the best explanation' because it starts from the observed facts and probes backwards into the reasons or explanations for these facts (Walton, 2001). Therefore, it also supports the transition from conjecturing to proving modality (Peirce, 1960; Arzarello et al., 1998). Abductive argumentation was used to analyze students' interactions and proving styles while they were discovering mathematical knowledge or generating the ideas of a proof. Therefore, it supports explanatory conjectures and the subsequent related proof. In geometry, proofs are normally deductive, but the discovering and conjecturing processes is often characterized by abductive argumentation. Particularly, in a dynamic geometry environment like GeoGebra, the produced data might sow the seeds of generating abductive argumentation. Its strength depends on all evidence and data which are collected by dragging, observing, measuring, conjecturing, and checking the relationship between the objects. It also might conduct the transition from exploring-conjecturing to validating-proving modality. Therefore, abductive argumentation is used to explore data, find and choose a pattern, and explain plausible hypotheses, which aim to determine the methods of solving problems and producing arguments for proofs.

4. Data analysis and results

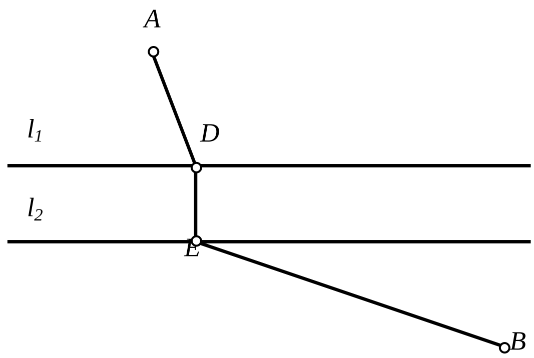
In order to analyze student's thinking and behavior during the proving process, we used an effective analysis method, called "frame analysis method" (see e.g. McDougall & Karadag, 2008) combined with an audio-taped method, to monitor and track the student's proving process without distracting the student while she/he was working on her/his tasks with the IHS. The method was based on recording student's manipulation and discussion in the computer environment by using a screen-casting Wink® software* and allowed us to capture not only what, but how she/he did on the dynamic GeoGebra worksheets. All students were required to formulate conjectures and write a

* This software also allowed us to zoom into any frame recorded and to annotate it. It also made the communication easier because we can easily navigate the frames and describe the moment of action in order to provide opportunity of just-in-time commenting.

formal proof. The IHS provided the students with a scaffold to bridge the gap between argumentation and proof and to realize geometric invariants. We analyzed students' task-based activities by comparing each frame* of record and tracking every movement of mouse and entry of keyboard. The recording software was set to record one frame per two seconds. After collecting the data, all of the videos, audio clips and snapshots were watched and listened to several times, so as to understand the students' thinking and behavior while they used the IHS to support proving activities. Students' discussions in these materials were annotated, transcribed on paper and finally translated into English. We also used Toulmin model of argumentation as a tool to analyze continuity and structural gaps between argumentation and proof. We have designed a group-based task (see one-bridge problem below) to observe some acts of proving, students' behavior as well as their interactions when they were using the IHS.

One-Bridge Problem. *A river has straight parallel sides and cities A and B lie on opposite sides of the river. Where should we build a bridge in order to minimize the traveling distance between A and B (a bridge, of course, must be perpendicular to the sides of the river)?*

We chose the discussion of one typical group of students to analyze the role of abductive argumentation during the proving process by using Toulmin model. This process provided us with interpreting the gap between abductive argumentation and deductive proof. The discussion was transcribed based on captured snapshots and audio clips as follows:

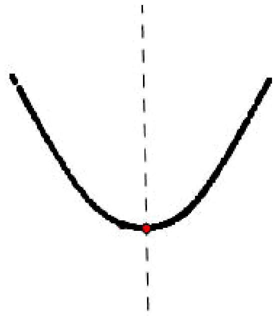
♣03. Student 2: Now we draw two parallel lines representing two banks of the rivers and then determine the position where we can build the bridge! ♣05. Student 1: Hey, these lines are not parallel! You move a point on the one line and look two lines. I think they are no longer parallel! ♣06. Student 3: That's right! I think you must use the parallel function of GeoGebra to construct these lines. ♣08. Student 2: But how can we know where point *D* should be situated? ♣10. Student 3: You can measure the length of sum the (*AD* + *DE* + *EB*) and observe the figure until the sum has a minimal value! ♣11. Student 1: I agree with you. ♣13. Student 2: Drag the point slowly please! In my opinion, this point is the position where we can build the bridge! ♣14. Student 1: Let me see. Yes, the red point which represents the sum (*AG* + *GH* + *HB*) is in the minimum point of the parabola. But what are the special characteristics in this case? ♣16. Student 3: Yeah, it is too difficult to see anything at the moment! ♣19. Student 2: I think these two lines seem to be parallel? Look at the figure! ♣20. Student 3: You can check it again by moving point *A*, point *B* or both to the new positions!	Students read information and requirements of the task. This group had an idea to model the situation but they had a habit of drawing a figure in the paper and pencil environment. Thus, they did not use the parallel function of GeoGebra and they drew two arbitrary lines that seem to be parallel. After moving a point they realized that they failed in modeling the situation and they had to use the *Construction level* in the IHS as follows: *- Draw two parallel lines representing two banks of the river using the parallel function of GeoGebra.* *- Draw two points A and B representing two cities.* *- Draw movable point D on the straight line* l_1. *- Draw a straight line passing through point D and perpendicular to the straight line* l_1*, cut the straight line* l_2 *at a point E. Construct three segments AD, DE, and EB.*  Students dragged point *D* but cannot determine where the distance from city *A* to

* A frame is defined as the snapshots of the computer screen at a specified moment.

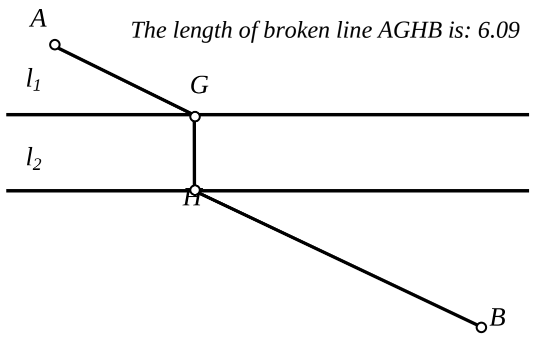
♣23. Student 1: Yes, the situation is the same in the new case! I think we have one more sub-invariant in this problem: *When the length of the broken line AGHB is minimal, two straight lines AG and HB are always parallel.*
♣24. Student 2: That's right! We have also two parallel lines representing two banks of the river, they are fixed lines; the points *A* and *B* are also fixed, therefore the distances from *A* and *B* to the lines l_1, l_2 are constant numbers. Can you see something more?
♣25. Student 1: But is it more important now, to identify what kind of transformations based on realized invariants we should use to solve this problem? They are line reflection, point reflection, translation or rotation?
♣26. Student 3: I think the transformation in this case is line reflection but which line is chosen as a line of reflection? And how can we explain the two parallel lines l_1 and l_2 under a line reflection?
♣27. Student 2: Exactly! These two lines cannot be images of each other under a line reflection. But they also can be images of each other under a translation!
♣28. Student 1: That is a reasonable argument but how we can determine the vector of this translation?
♣29. Student 2: You can imagine that if the line l_1 moved a distance towards the line l_2, you will realize the vector of translation. In my opinion, this vector has a length equal to the distance between two banks of the river (vector $\overrightarrow{HG}$).
♣30. Student 3: So we have: *The line l_1 is an image of the line l_2 under the translation of vector* $\overrightarrow{HG}$. But how can we construct point *G*?
♣31. Student 1: Since $T_{\overrightarrow{HG}}(HB) = AG$ point *G* must have lain on the line $T_{\overrightarrow{HG}}(AG)$ passing through point *A* and point $B' = T_{\overrightarrow{HG}}(B)$.
♣32. Student 2: So now we have to prove that *G*, *H* are two points we can build two ends of the bridge. It means that the distance from *A* to *B* passing through the points *G*, *H* is minimal.
♣33. Student 3: It is obvious that the length of the broken line *AGHB* is smaller the length of broken line *ADEB*. How can we prove this inequality?
♣34. Student 1: We already have the following data: *HG* = *BB'* = *ED* = constant; *HB* = *GB'*, *DB'* = *ED* and *HB* // *AB'*, etc.
♣38. Student 3: We will start from the inequality:

$$AG + GH + HB \leq AD + DE + EB \qquad (1)$$

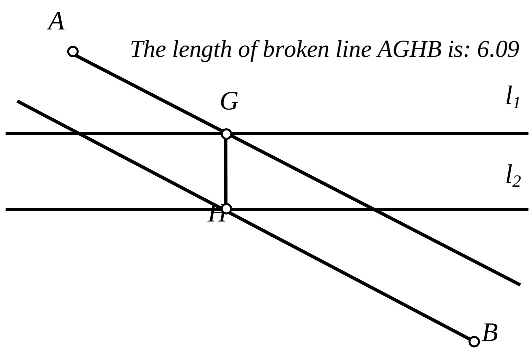
city *B* minimal is.



In order to justify this hypothesis, students used the *Invariance level* in the IHS:
- *Draw two straight lines passing through A, G and H, B. You can change the position of points A, B and realize the invariants.*
- *Write your realized invariants on a piece of paper.*



Students named the points, whose lengths are minimal, *G* and *H*. They moved point *G* to and fro many times but they could not see anything. Finally, they decided to use the *Conjecture level* in the IHS:
- *What is the relationship between two lines AG and HB?*
- *Write your conjectures on a piece of paper.*



Students formulated a conjecture: *If two lines AG and GB are parallel then the sum of broken line AGHB is minimal.*

But how can we prove this inequality?

♣40. Student 1: We use the collected data to derive:

$AF + FG + GB = AF + BB' + FB'$ (2)

$AD + DE + EB = AD + DB' + BB'$ (3)

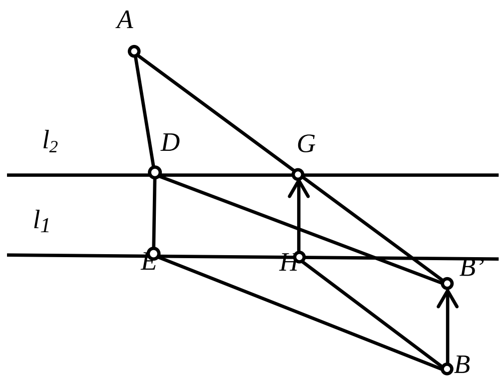
♣41. Student 2: Look! We have *BB*' as a common summand, so we need only to prove that:

$AF + FB' = AB' \leq AD + DB'$ (4)

♣44. Student 3: That is a triangle inequality! So now we can write a formal proof.

♣47. Student 2: But where can we start to prove this problem?

♣48. Student 1: I think we must construct point *B*', point *G* and point *H*. After that we can derive the target inequality (4) from the departing inequality (1).



Students wrote a formal proof as follows:

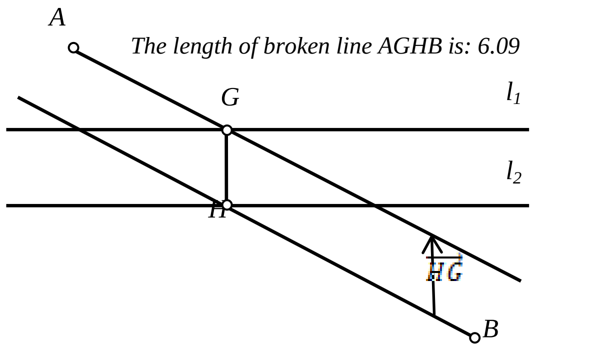
Let $B' = T_{\overrightarrow{HG}}(B)$ *and* $G = AB' \cap l_1$. *We have following equalities:*

$ED = HG = BB'; EB = DB', HB = GB'$

Therefore: $AD + DE + EB = (AD + DB') + BB'$

$\geq AB' + BB' = AF + FG + GB$

Equality occurs when only when three points A, D, B' are collinear.



We realized that some groups of students could not discover geometric invariant, so they could not solve the problem. Therefore, recognizing invariant is one of the most important phases in the proving process. Students formulated one more conjecture: *The line* l_1 *is image of the line* l_2 *under the translation of vector* $\overrightarrow{HG}$. Then, they used abductive argumentation so as to find a way to construct point *G*. In the next step, students had difficulties in validating a conjecture. Thus, they use the *Argumentation level* in the IHS as follows:

- *Compare the length of broken line ADEB and broken line AGHB.*
- *Write all of your arguments on a piece of paper.*

The following steps are abductive argumentation:

C_1: $ED = HG = BB'$; $HB = GB'$ and $EB = DB'$

D_1 = ? ⟶ **C_1**

W_1: Property of translation

D_1: $E = T_{\overrightarrow{HG}}(D)$; $B' = T_{\overrightarrow{HG}}(B)$; and $H = T_{\overrightarrow{HG}}(G)$

C_2: $AG + GH + HB \leq AD + DE + EB$

D_2 = ? ⟶ **C_2**

W_2: C_1

D_2 = C_3: $AG + GB' + B'B$
$\leq AD + DB' + B'B$

D_3 = ? ⟶ **C_3**

W_3: *BB*' is common summand

D_3 = C_4: $AG + GB' \leq AD + DB'$

D_4 = ? ⟶ **C_4**

W_4: *A*, *G*, *B*' are collinear

The conclusion **C_4** of the previous step is the data needed to apply the inference to the next step.

D_4 = C_5: $AB' \leq AD + DB'$

D_5 = ? ⟶ **C_5**

W_5: Triangle inequality

D_5 is a mathematical theorem.

At the *proof level*, students combined some produced arguments in order to write a formal proof. Finally, at the *delving level*, they expanded the problem or found the other shorter solutions.

In this research, students followed some open-ended questions and explorative tasks in the IHS. These helps are necessary factors during the proving process, e.g., for the recognition of geometric invariants, the exploration of the problem situation, and the collection of additional data (especially some valid arguments). In the snapshots, the first explorative phase (including searching for invariants, formulating conjectures) consists mainly of constructing drawings, measuring the segments or angles, checking the relationships between figures, lines, angles, etc. The students spent almost all of their time on *invariance*, *conjecture*, *argumentation*, and *proof* phases. In this analysis, the proving process was separated into chucks, which include different levels of proving. We analyzed based upon students' discussion and movements on the screen. We considered any period of time with "no change" or "no sound" as silence time (or thinking).

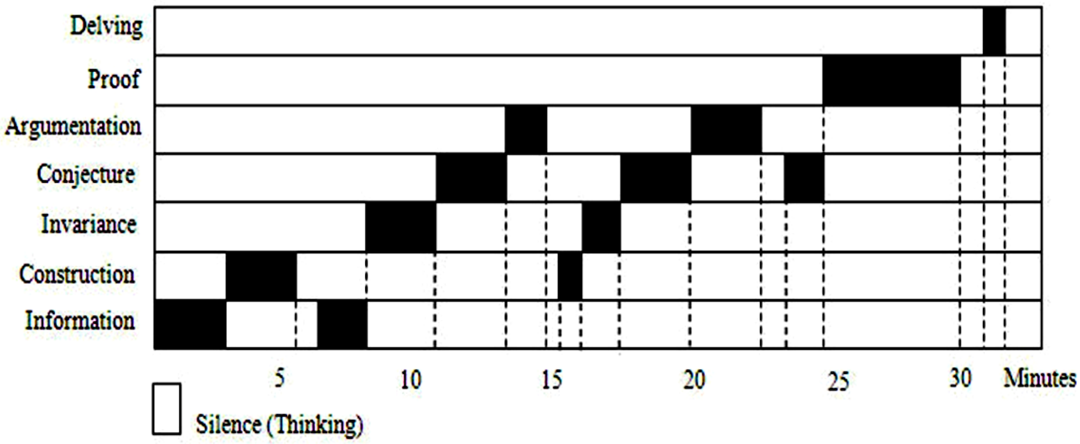


Fig. 5: Time-line graph of the proving process in the one-bridge problem

In the two-bridge problem, we have realized that some students *reversed abductive structure* (see Fig. 6 below) in order to write formal proofs. They started from a mathematical theorem and found data D_5 for validating claim C_5, found data D_4 for validating claim C_4, and so on. This strategy would be the best way to support the students in understanding the meaning of proving activities and understanding formal proofs as well.

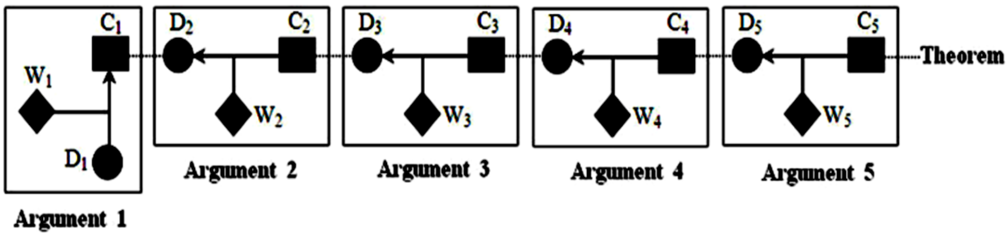


Fig. 6: Abductive structure of argumentation in the two-bridge problem

During the process of working with one-bridge problem, students try to transform abductive argumentation into deductive proof. However, there were some students who could not cover the structural gap between abductive argumentation and deductive proof. The strength of the deductive chain seems to be so strong that they are not able to construct continuity in the referential system because words and expressions used in the argumentation and proof are often of the same format (Pedemonte, 2001). Therefore, we tried to describe the structure of abductive argumentation in order to interpret the way students combine their valid arguments.

Consequently, we realized a gap between the structures of the two processes because some students could not combine collected arguments into a logical way. However, with the support of the IHS, students could overcome this difficulty in order to write a formal proof.

Departing from the different phases of the proving process and the structure of argumentation, we also proposed three conditions for understanding the development of the proving process. These conditions may provide an instrument for determining whether or not students understand the proving process within a dynamic geometry environment:

- *Realizing the geometric invariants for generating ideas for proofs.* This is an important phase in the development of the proving process. Realizing geometric invariants supports students in getting more data for proving and searching for the ideas of proofs by using geometric transformations.
- *Understanding the relationship between argumentation and proof.* Students have difficulties in constructing proofs and fail in writing proofs because they do not determine the continuity and gap between argumentation and proof. Therefore, understanding this relationship helps students to bridge the gap and effectively utilize the continuity between them. This continuity helps students to produce arguments in the transition from conjecture to proof. These arguments were selected from a set of spontaneous arguments. This understanding also includes the ability of using different kinds of inferences during the proving process such as deduction, induction, and abduction.
- *Organizing arguments in order to write a formal proof.* This is one of the most difficult phases in the proving process because students have to use formal language, symbols, and notations. They also need to combine and organize produced arguments as a chain of logical arguments to form a formal proof.

On the basis of these conditions, some open-ended questions and explorative tasks in the IHS were designed aimed to support students during the proving process. We have also formulated a hypothesis to verify the influence of the IHS on increasing student's proving level after the period of experimental teaching. This hypothesis was presented as follows: "*There is no significant difference between students' level of proving before and after the period of experimental teaching*". In order to test the hypothesis, we recorded the students' level of proving in the pre-test and post-test (67 students in the experimental group) and determined whether the two sets of levels of proving come from the same distribution. To carry out this work, the one sample Kolmogorov-Smirnov test was used. It took the observed cumulative distribution of levels of proving and compared them to the theoretical cumulative distribution for a normally-distributed population.

		Proving level (pre-test)	Proving level (post-test)
N		67	67
Normal Parameters[a,,b]	Mean	3.73	4.27
	Std. Deviation	.963	1.009
Most Extreme Differences	Absolute	.207	.214
	Positive	.179	.164
	Negative	-.207	-.214
Kolmogorov-Smirnov Z		1.694	1.748
Asymp. Sig. (2-tailed)		.006	.004

a. Test distribution is Normal

b. Calculated from data

Table 1: One-sample Kolmogorov-Smirnov test

From table 1, we have obtained the following results: $Z = 1.694$, $p < 0.05$ (in the pre-test) and $Z = 1.748$, $p < 0.05$ (in the post-test). This indicates the observed distribution corresponds to a theoretical distribution. That is, the data are not significantly different to a normal distribution at the $p < 0.05$ level of significance. Furthermore, we used the Wilcoxon signed-ranks test to analyze the initial situation. This test is the nonparametric equivalent of the related t test.

		N	Mean Rank	Sum of Ranks
Proving level (post-test) & Proving level (pre-test)	Negative Ranks	15[a]	21.80	327.00
	Positive Ranks	36[b]	27.75	999.00
	Ties	16[c]		
	Total	67		

a. Proving levels in the post-test < Proving levels in the pre-test

b. Proving levels in the post-test > Proving levels in the pre-test

c. Proving levels in the post-test = Proving levels in the pre-test

Table 2: Proving levels ranks in the experimental group

	Proving levels (post-test) & Proving levels (pre-test)
Z	-3.269[a]
Asymp. Sig. (2-tailed)	.001

a. Based on negative ranks

b. Wilcoxon Signed Ranks Test

Table 3: Wilcoxon test statistics[b]

The results of the Wilcoxon test are as follows: Z = -3.269, N = 67, $p < 0.01$. Therefore, we can conclude that the students' levels of proving do differ significantly after the treatment. There are 36 students increasing their levels of proving, 15 students decreasing their levels while 16 students having the same levels in the post-test. In general, this result shows the positive effect of the IHS in the proving process and that the methodological model really improves the student's level of proving. For that reason, teachers should design such a methodological model in the mathematics classroom in order to provide their students a strategy for proving and improve proving levels as well.

5. Conclusions and recommendations

This paper classifies student's proving level and provides a methodological model for understanding the development of the proving process within a dynamic geometry environment. This model relates to three basic conditions for understanding the proving process and refers to some fundamental aspects during the process of constructing formal proofs. The essence of learning proofs is to understand the proving process and use appropriate strategies and tools as a means of exploration, discovery, and invention. For that reason, there should also be a distinction between understanding a *proof as product* and a *proof as process*. For tertiary students, in order to realize the proof ideas in solving open problems, they need to understand the development of the proving process. This understanding does not solely consist of knowing how each phase logically follows the previous phases. It includes the process of producing arguments during conjecture validation and the transition from argumentation to proof. In addition, throughout our experimental teaching, different difficulty categories that students met during the proving process were also revealed such as *lack of explorative strategy*; *difficulty in reading diagram*, *producing arguments*, and *organizing valid arguments to write a formal proof*. These difficulties also show the gap between conjecture and proof. In other words, students face with difficulty in producing 'valuable arguments' for proofs. They could not differentiate valuable arguments from a set of collected arguments and could also not reverse the structure of abductive argumentation in order to write a deductive proof. However, with the support of the IHS, some students can overcome these difficulties in writing proofs. We have also revealed that realizing geometric invariants is a crucial element in generating ideas for proofs. Furthermore, we have realized that *dynamic visual thinking* supports students in realizing geometric invariants. Thus, developing students' dynamic visual thinking would provide them with the ability to observe the static and moving invariants, realize the properties of shapes, interpret the diagram, and transform from the diagram into a chain of arguments. In particular, it has also improved students' logical skills by linking between dynamic visual diagrams and formal arguments. Indeed, this kind of thinking provides students with 'a vision' of realizing geometrical facts, internalizing specific facts, learning to reason, shifting attention from specific relationships to properties, and then reasoning on the basis of realized and then perceiving properties. Through these activities students' powers are engaged and developed, such as the power to imagine, to express what is imagined in figures, diagrams, movements, invariants, and symbolic objects. These powers are emphasized both before and after using dynamic geometry software; especially the ability of imagining or re-imagining what changed and what invariants remained the same without dynamic geometry software.

During our experimental teaching, we also proposed three basic conditions for understanding the proving process. These conditions determine the *territory before a proof*, realize different levels of proving, realize geometric invariants for generating ideas for proofs, construct a cognitive unity in the transition from conjecture to proof, understand the relationship between argumentation and proof, use different kinds of argumentation during the proving process, and organize arguments to write a formal proof. It means that to understand the proving process, every student needs to know how to work and experience these conditions during the proving process with the support of dynamic geometry software. This dynamic environment creates collaborative activities to explore open problems and provides useful elements to explain why and how this tool can be a support for proving activities. Therefore, the findings of this research can be used to enhance the quality of learning and teaching proof both at the tertiary level and the secondary level.

In some countries, a lot of crucial aspects of mathematics including proofs and proving have been reduced in importance or eliminated from the mathematics curriculum and basic requirements of secondary school. However, we thought that proofs and proving have also played an important role in the curriculum of mathematics in secondary schools, especially in upper secondary school. Through proving activities, students can realize the meaning of mathematics in their real-world life. Therefore, they should understand the development of the proving process not only in the geometry field but also in other fields of mathematics. In addition, in the teaching of proof in secondary schools, mathematics teachers should focus on the pedagogical tasks which contain typical mathematical processes to work in depth with. Teachers should also provide students with a rich opportunity to make a conjecture during the proving process because once students have formulated a conjecture they often visualize and collect data from diagrams and figures to produce arguments. These arguments may provide a few clues for the proving process, although do not produce an instant proof. Therefore, it is necessary to design visualizing activities in order to support students in transforming from figural into conceptual aspect during the proving process.

References:

1. Arzarello, F., Micheletti, C., Olivero, F., Paola, D., & Robutti, O. (1998). A model for analyzing the transition to formal proofs in geometry. *Proceedings of the 22th PME International Conference*, Stellenbosh, South Africa, vol.02, pp.24-31.
2. Balacheff, N. (1998). Construction of meaning and teacher control of learning. In Tinsley D. J., Johnson D. C. (Eds), *Information and Communication Technologies in School Mathematics*, pp. 111-120. London: Chapman & Hall Publishing House.
3. Boero, P., Garuti, R., & Mariotti, M. A. (1996). Some dynamic mental processes underlying producing and proving conjectures. *Proceedings of the 20th PME International Conference*, Valencia, Spain, vol.02, pp.121-128.
4. De Villiers, M. (2003). *Rethinking proof with sketchpad*. California: Key Curriculum Press.
5. Edwards, L.D. (1997). Exploring the territory before proofs: students' generalizations in a computer microworld for transformation geometry. *International Journal of Computers for Mathematical Learning, 02*, 187-215.
6. Hanna, G. (2000). Proof, explanation and exploration: an overview. *Educational Studies in Mathematics*, *44*, 05-23.
7. Hohenwarter, M., & Jones, K. (2007). Ways of linking geometry and algebra: the case of GeoGebra. *Proceedings of the British Society for Research into Learning Mathematics*, *27*(03), 126-131.

8. Magnani, L. (2001). *Abduction, reason, and science: processes of discovery and explanation*. New York: Kluwer Academic Publishers.

9. Mariotti, M. A. (2007). Proof and proving in mathematics education. In A. Guitierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education*. Rotterdam: Sense Publishers.

10. McDougall, D., & Karadag, Z. (2008). Tracking students' mathematical thinking online: frame analysis method. *Proceedings of the 11th International Congress on Mathematical Education*, Monterrey, Nuevo Leo, Mexico.

11. Olivero, F. (2002). The proving process within a dynamic geometry environment. *A Dissertation for the Degree of Doctor of Philosophy in Mathematics Education*, University of Bristol, England.

12. Pedemonte, B. (2001). Relation between argumentation and proof in mathematics: cognitive unity or break?. *Proceedings of the 2nd Conference of the European Society for Research in Mathematics Education*, pp.70-80.

13. Pedemonte, B. (2007). How can the relationship between argumentation and proof be analyzed. *Educational Studies in Mathematics, 66*, 23-41.

14. Pedemonte, B., & Reid, D. (2011). The role of abduction in proving processes. *Educational Studies in Mathematics, 76*, 281-303.

15. Peirce, C. S. (1960). *Collected papers of C. S. Peirce*. Massachusetts: Harvard University Press.

16. Polya, G. (1954). *Mathematics and plausible reasoning*. New Jersey: Princeton University Press.

17. Polya, G. (1962). *Mathematical discovery: on understanding, learning, and teaching problem solving*. New York: Combined Edition, Wiley.

18. Toulmin, S. (1958). *The uses of argument*. Cambridge: Cambridge University Press.

19. Walton, D. (2001). Abductive, presumptive and plausible arguments. *Informal Logic, 21*, 141-169.

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Развитие доказательного процесса в динамичной геометрической среде

Данх Нам Нгуйен

Педагогический Университет Тхай Нгуен, Вьетнам
E-mail: danhnam.nguyen@dhsptn.edu.vn

Аннотация. В данной работе классифицируется доказательный уровень студентов и создание интерактивной справочной системы, соответствующей этим уровням с целью изучения развития доказательного процесса в динамичной геометрической среде. Данная справочная система также использовалась для обеспечения студентов вуза стратегией доказательства и улучшения их доказательного уровня. Вопросы, допускающие бесконечное множество ответов и исследовательские задачи в данной системе содействуют изучению доказывания студентами, особенно в процессе реализации инвариантов, формулировки гипотез, представления доводов и написании доказательств. В данном исследовании делается попытка повлиять на известные трудности студентов при написании формальных доказательств. В работе делается предположение, что эти трудности основываются на недостатке понимания студентами отношений между аргументацией и доказательством. Поэтому, чтобы проанализировать структуру аргументации студента и изучить роль силлогизма с вероятной малой посылкой в написании дедуктивного доказательства, мы использовали модель аргументации Тулмина. Более того, данная работа представляет учителям математики три основных условия для понимания развития доказательного процесса и стратегий обучения для содействия студентам в построении формальных доказательств.

Ключевые слова: доказательство; доказательный процесс; доказательный уровень; интерактивная справочная система; модель Тулмина; аргументация; силлогизм с вероятной малой посылкой; образное мышление; геометрический инвариант.